PROBLEMS

LECTURE POSTECH 19 AUGUST, 2014
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1. worm

Problem 1.1. Let Ω be the worm domain. Show that every strongly pseudoconvex boundary point can be exposed.

This problem was solved in [3] for domains which are strongly pseudoconvex. See [2] for definition of the worm . The worm is strongly pseudoconvex except on an annulus. It has no Stein neighborhood basis. But you can still solve the equation $\overline{\partial} u = f$ on the closure, so that u is $\mathcal{C}^\infty$ if f is $\mathcal{C}^\infty$, see Kohn, [10].

2. Nirenberg problem

Problem 2.1. (Nirenberg) Let U be a strongly pseudoconvex bounded domain in $\mathbb{C}^n$ with smooth boundary. Suppose that γ is a smooth curve in the boundary which is transverse to the complex tangent space at each point. Can it happen that there exists a continuous function f om $\overline{U}$ holomorhic on U such that f vanishes identically on γ but does not have any zero inside U.

Note, the problem is that γ is not assumed to be real analytic. If it is real analytic it extends as a complex curve to the inside. Then f must be zero there.

3. dbar

Problem 3.1. Solve $\overline{\partial} u = f$ with supnorm estimates on bounded convex domains in $\mathbb{C}^2$ with $\mathcal{C}^\infty$ boundary.

If the boundary is real analytic, one can solve dbar with supnorm estimates. The problem is that there might be infinitely flat points.

4. Bergman Space

Problem 4.1. Let U be an unbounded (smooth strongly) pseudoconvex domain in $\mathbb{C}^2$. If $A^2(U) \neq \{0\}$, is $A^2(U)$ infinite dimensional. (A^2 consists of L^2 integrable holomorphic functions on U.)

Date: August 20, 2014.

5. Polynomial Convexity

Problem 5.1. Let X be a complex hypersurface in $\mathbb{C}^3$ with an isolated normal singularity at 0. Suppose $K \subset X \setminus \{0\}$ is compact. Suppose that 0 is in the polynomially convex hull of K. Let $K \subset F \subset X$ be contained in the relative interior. Is 0 in the relative interior of the polynomial hull of F.

This problem originated in some questions about the Levi problem in complex spaces, [7]

6. Complex Dynamics/Real dynamics

Problem 6.1. Let $P(z)$ be a complex polynomial in $\mathbb{C}$. For $z \in \mathbb{C}$ let $z_n = x_n + iy_n = P^n(z)$. We call the sequence $\{x_n\}$ the real part of the orbit. Suppose one only knows the real orbits. How much can one say about the complex dynamics of P? For example, how can one detect the degree of P?

See paper by F-Peters [8]. Almost nothing is done on this kind of problem. In general one can iterate a map $F : \mathbb{R}^k \to \mathbb{R}^k$ and one only can see the parts of the orbits, $(x_1^n, \dots, x_\ell^n)$ where $\ell < k$. What can one then say about the dynamics.

7. Complex Henon maps

Introduce complex Henon maps, $H(z, w) = (z^2 - aw, bz)$. Need $K^\pm, J^\pm, J$ J is possible definition of Julia set. Also $G^\pm$. Alternative definition, $\mu^\pm = dd^cG^\pm$. $\mu = \mu^+ \wedge \mu^-$. Main theorem in field is that the measure μ is the unique invariant probability measure of maximal entropy. In one variable, the support of this measure equals the Julia set. Therefore one has the question:

Problem 7.1. Let H be a complex Henon map. Is $J^* = J$? [5]

8. Fatou-Bieberbach domains

Definition of Fatou- Bieberbach (FB) domains. They are domains in $\mathbb{C}^2$ which are biholomorphic to $\mathbb{C}^2$ while being proper subsets. (Same in $\mathbb{C}^n, n > 2$.)

Standard construction of Fatou-Bieberbach domains, see Rosay-Rudin [12] . Take a biholomorphic self map of $\mathbb{C}^2$. For example a Henon map H with $a, b < 1$. Then there is a small ball B so that if $(z, w) \in B$ then $\|H(z, w)\| \leq c\|(z, w)\|$ on B, $c < 1$ some constant. This means that on B, $\|H^n(z, w)\| \to 0$. We say that B is contained in the basin of attraction of 0. Let Ω denote the set of all points such that $H^n(z, w) \to 0$. This is an open set and it is biholomorphic to $\mathbb{C}^2$. One also sees that if we start with $(z, w) = (100, 0)$ then $H^n(z, w) \to \infty$. Hence this point is not in Ω. Hence this is a Fatou Bieberbach domain.

Note that since a Fatou-Bieberbach domain is biholomorphic to $\mathbb{C}^2$, then it contains a smaller FB domain. In fact we can find a sequence of FB domains $\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_n \supset ....$ One can then ask if it is possible to find such a sequence such that $\cap \Omega_n = \emptyset$. If this is true, then an old conjecture by Michael is true: All characters on a Frechet algebra are continuous. (See Dixon-Esterle [4] for precise statements)

One can have Fatou-Bieberbach domains V with a boundary which is $\mathcal{C}^\infty$, Stensønes, [13]

Problem 8.1. The boundary of V is a union of Riemann surfaces. Are they all biholomorphic copies of $\mathbb{C}$? Does there exist an FB domain which has real analytic boundary?

Problem 8.2. (Hartz-Shcherbina-Tomassini, [9]) Does there exist an FB domain which is contained in a proper strongly pseudoconvex subdomain in $\mathbb{C}^2$?

One can perturb the construction of FB domains: Pick two numbers $0 < a < b < 1$. Let F_n be a sequence of biholomorphisms of $\mathbb{C}^2$ such that if $\|(z,w)\| < 1$ then $a\|(z,w)\| \leq \|F_n(z,w)\| \leq b\|(z,w)\|$. Then one can consider the iterates $F^{(n)} = F_n \circ \cdots F_1$. Let $\Omega = \{(z,w); F^{(n)}(z,w) \to 0$. We call this a uniformly random basin (or non-autonomous basin), see [1] for a recent survey.

Problem 8.3. Are uniformly random basins FB domains?

There are partial results if a and b are close together. (See Peters-Smit [11] on Arxiv recently)

Definition 8.4. Projective compactifiaction of a ball: $B_R = \{\|z\| < R\}$. Identify the boundary of B_R with the plane at infinity. This gives topology to the closure. The boundary and the inside both have complex structure. But they match poorly. Note that if we take a limit when we let $R \to \infty$ we get the usual $\mathbb{P}^n$.

Now consider a random basin Ω. Then Ω is an increasing union of balls, $B^n = \{(F^{(n)})^{-1}(B)\}$. Use this compactification. And try to pass to the limit.

Problem 8.5. Does such a limit exist and does it give a complex structure on a compactification of the random basin?

Remark 8.6. If the random basin is not biholomorphic to $\mathbb{C}^n$ then this gives a new complex structure to $\mathbb{P}^n$. If this can be done when $n = 3$ this provides a complex structure to S^6 (according to Siu)

Remark 8.7. An alternative to the uniformly random basins are obtained by removing the condition of the lower bound a. In such a case one can obtain random basins which are not boholomorphic to $\mathbb{C}^n$. (F- short $\mathbb{C}^n$, [6])

If one can use the above compactification in the case of short $\mathbb{C}^3$, then one surely gets a nonstandard $\mathbb{P}^3$.

Problem 8.8. Does the above compactification work for short $\mathbb{C}^3$?

Harz-Shcherbina-Tomassini, [9] has introduced the core of a domain. There is also a notion of core for short $\mathbb{C}^2$. [6]

Problem 8.9. Describe the core of a short $\mathbb{C}^2$.

Problem 8.10. Let Ω be a random basin. Show that there exists a proper holomorphic map from $\mathbb{C}$ into Ω. [There exist many non-constant holomorphic maps from $\mathbb{C}$ into Ω, see [1]]

This problem can be considered to be the next step in the process of showing that uniformly random basins are biholomorphic to $\mathbb{C}^n$.

Random basins occur as stable manifolds of hyperbolic maps on complex manifolds. One has similar questions.

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